Departement Physik, Universität Basel
Prof. C. Bruder (Zimmer 4.2, Tel.: 061 207 36 92, Christoph.Bruder@unibas.ch)

Classical and Quantum Nonlinear Dynamics
Frühjahrsemester 2024

Blatt 6

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Tutor: Parvinder Solanki Zi. 448; parvinder.parvinder@unibas.ch

(1) **Lorenz equations for large r** (5 Punkte)
Consider the Lorenz equations in the limit of $r \to \infty$, By taking the limit in a certain way, all the dissipative terms in the equations can be removed.

(a) Let $\epsilon = r^{-1/2}$. Find a change of variables such that as $\epsilon \to 0$, the equations become

$$\begin{aligned} X' &= Y \\ Y' &= -XZ \\ Z' &= XY \,. \end{aligned}$$

(b) Find two conserved quantites (i.e., constants of motion) for the new system.

(c) Show that the new system is volume-preserving.

(d) Explain physically why the Lorenz equations might be expected to show some conservative features in the limit $r \to \infty$.

(e) Solve the new system numerically. What is the long-term behavior? Does it agree with the behavior seen in the Lorenz equations?

(2) **Logistic parabola** (5 Punkte + 3 Bonuspunkte)
The logistic map

$$x_{n+1} = r x_n (1 - x_n)$$

is a discrete-time analog of the logistic equation of population growth and one of the most studied mathematical "laboratories" for chaos. The goal of this problem is to write a program to plot its orbit diagram, i.e., reproduce Fig. 10.2.7 of Strogatz that has become an icon of nonlinear dynamics.

(a) Write a computer program with two loops. For each $r \in [2.5, 4]$, generate an orbit $x_0, x_1, x_2, \ldots$ starting from some random initial condition $x_0 \in [0, 1]$. Iterate for 300 cycles or so, to allow the system to settle down to its eventual behavior. Once the transients have decayed, plot many points, say $x_{301}, \ldots, x_{600}$ above that r. Then move to the next value of r and repeat.

(b) Show that near $r \approx 3.83$ there is a stable period-3 cycle. Show that there is a stable period-5 cycle near $r \approx 3.9057065$.

(c) Convince yourself that the orbit diagram is self-similar by plotting the window $r \in [3.847, 3.857]$ and restricting the vertical axes to values in $[0.13, 0.175]$, see lower panel of Fig. 10.2.7.

(d) Amazingly, the dynamics calculated in (a) is *universal* for all unimodal maps (i.e., maps for which the RHS is smooth, concave down and has a simple maximum). Repeat (a) for the sine map

$$x_{n+1} = r \sin \pi x_n$$

for $r \in [0, 1]$ and $x \in [0, 1]$. Plot the orbit diagram and compare to the logistic map.