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Classical and Quantum Nonlinear Dynamics
Frühjahrsemester 2024

Blatt 1

Abgabe: 07.03.2024 12:00
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(1) **Linear stability analysis** (1 Punkt)
Use linear stability analysis to classify the fixed points of the following systems. If linear stability analysis fails because of $f'(x^*) = 0$, use a graphical argument to decide the stability: (a) $\dot{x} = ax - x^3$ (b) $\dot{x} = 1 - \exp(-x^2)$ (c) $\dot{x} = \tan x$.

(2) **Fixed points** (2 Punkte)
For each of (a)-(e), find an equation $\dot{x} = f(x)$ with a smooth f, or if there are no examples, explain why not: (a) Every real number is a fixed point. (b) Every integer is a fixed point, and there are no others. (c) There are precisely three fixed points, and all of them are stable. (d) There are no fixed points. (e) There are precisely 100 fixed points.

(3) **"Blow-up": Reaching infinity in a finite time** (2 Punkte)
Show that the solution to $\dot{x} = 1 + x^{10}$ escapes to $+\infty$ in a finite time, starting from any initial condition.
Hint: Don't try to find an exact solution; compare the solutions to those of $\dot{x} = 1 + x^2$.

(4) **Infinitely many solutions with the same initial condition** (2 Punkte)
Show that the initial value problem $\dot{x} = x^{1/3}$, $x(0) = 0$, has an infinite number of solutions.
Hint: Construct a solution that stays at $x = 0$ until some arbitrary time t_0, after which it takes off.

(5) **Phase portrait of the van der Pol oscillator** (3 Punkte)
The van der Pol oscillator described by the equation of motion

$$\ddot{x} + \eta\dot{x}(-1 + x^2) + x = 0 \tag{1}$$

is a paradigmatic example of a nonlinear oscillator.

(a) Interpret the terms in (1).

(b) Use a computer to plot the phase portrait of the van der Pol oscillator: i.e., for a fixed value of η, plot $\dot{x}$ over x for a set of typical initial conditions $(x(0), \dot{x}(0))$. Discuss your results for different η and compare to the case of a harmonic oscillator ($\eta = 0$).

(6) **The leaky bucket (abbreviated from Strogatz, prob. 2.5.6)** (3 Bonuspunkte)
The following example shows that in some physical situations, non-uniqueness is natural and obvious, not pathological.
Consider a water bucket with a hole in the bottom. If you see an empty bucket with a puddle beneath it, can you figure out when the bucket was full? No, of course not. It could have finished emptying a minute ago, ten minutes ago, or whatever. The solution to the corresponding differential equation must be non-unique when integrated backwards in time.
Here's a crude model of the situation. Let $h(t)$ = height of the water remaining in the bucket at time t; a = area of the hole; A = cross-sectional area of the bucket (assumed constant); $v(t)$ = velocity of the water passing through the hole.

(a) Show that $av(t) = A\dot{h}(t)$. What physical law are you invoking?

(b) Use conservation of energy to derive an additional equation, viz., $v^2 = 2gh$, where g is the gravitational acceleration.
Combine with (a) to obtain $\dot{h} = -C\sqrt{h}$ where $C = \sqrt{2g}(a/A)$.

(c) Given $h(0) = 0$ (bucket empty at $t = 0$), show that the solution for $h(t)$ is non-unique in backwards time, i.e., for $t < 0$.